Summary: Itô's formula
Exercises

Itô's diffusions: $\{X(t), t \geq 0\}$ is a diffusion if it is solution of the following stochastic differential equation:

(*) $$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t)$$

where $\{W(t), t \geq 0\}$ is the Brownian motion

(*) $\Leftrightarrow$ $$X(t) = X(0) + \int_0^t \mu(s, X(s))\,ds + \underbrace{\int_0^t \sigma(s, X(s))\,dW(s)}_{\text{Itô's integral}}$$

$\rightarrow E\left[\int_0^t f(s,X(s))\,dW(s)\right] = 0$

$\rightarrow E\left[\left(\int_0^t f(s,X(s))\,dW(s)\right)^2\right] = \int_0^t E\left[f^2(s,X(s))\right] ds$

Question: Imagine that you have $\{X(t), t \geq 0\}$ given by (*) and now you want to compute

$$d\, f(t, X(t)) = ?$$

In the "usual" way, we use chain rule. Here that won't be the case!! So we next provide "new" rules to compute derivatives of functions.

table of infinitesimals:

$$(dt)(dt) = 0 \quad \Rightarrow (dt)^k = 0,\ \forall k \geq 2$$
$$(dW(t))(dW(t)) = dt$$
$$(dW(t))(dt) = 0$$

Example of these formulas:

$$dx(t) = \mu_x x(t)\,dt + \sigma_x x(t)\,dw(t)$$
$$dy(t) = \mu_y y(t)\,dt + \sigma_y y(t)\,dw(t)$$

$$[dx(t)][dy(t)] = \mu_x \mu_y x(t) y(t) \cancel{(dt)^2}$$
$$+ \sigma_x \mu_y x(t) y(t) \cancel{dw(t)\,dt}$$
$$+ \mu_x \sigma_y x(t) y(t) \cancel{dt\,dw(t)}$$
$$+ \sigma_x \sigma_y x(t) y(t) \underbrace{(dw(t))^2}_{dt}$$

ie:

$$[dx(t)][dy(t)] = \sigma_x \sigma_y x(t) y(t)\,dt$$

$$\Leftrightarrow \quad \frac{dx(t)\,dy(t)}{x(t)\,y(t)} = \sigma_x \sigma_y\,dt$$

$\sigma_x \sigma_y$ is a deterministic component, meaning that

$\left\{ \frac{dx(t)\,dy(t)}{x(t)\,y(t)},\ t \geq 0 \right\}$ is a deterministic process.

Itô's formula: Let $\{x(t), t \geq 0\}$ be a process as before, meaning that is such that:

$$dx(t) = \mu(x(t))\,dt + \sigma(x(t))\,dw(t)$$

Now let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that is $C^1 \times C^2$

$$f(t, x(t))$$

we may compute $\frac{\partial f}{\partial t}$

we may compute $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$

Then:

$$df(x, X(\tau)) = \left[\frac{\partial f}{\partial \tau} + \mu(X(\tau))\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X(\tau))\frac{\partial^2 f}{\partial x^2}\right] d\tau$$

$$+ \sigma(X(\tau))\frac{\partial f}{\partial x}\, dW(\tau)$$

Example: $\{S(\tau), \tau \geq 0\}$ is a geometric Brownian motion:

$$S(\tau) = S(0)\, e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma W(\tau)} \qquad S(0) = 1$$

$dS(\tau) = ?$ (by using Itô's formula)

"initial process" $dX(\tau) = dW(\tau)$ $\quad \mu(X(\tau)) = 0$, $\sigma(X(\tau)) = 1$

"function" $f(\tau, x) = e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma x}$

$$\left[f(\tau, X(\tau)) = f(\tau, W(\tau)) = e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma W(\tau)} \right]$$

$$\frac{\partial f}{\partial \tau} = (\mu - \tfrac{1}{2}\sigma^2)\, e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma x} = (\mu - \tfrac{1}{2}\sigma^2) f(\tau, x)$$

$$\frac{\partial f}{\partial x} = \sigma\, e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma x}$$

$$\frac{\partial^2 f}{\partial x} = \sigma^2\, e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma x}$$

Now we use Itô's formula:

$$d\left[e^{(\mu - \frac{1}{2}\sigma^2)\tau + \sigma W(\tau)} \right] = dS(\tau)$$

$$= \left[(\mu \cancel{- \tfrac{1}{2}\sigma^2})\, S(\tau) + 0 + \cancel{\tfrac{1}{2}\, 1 \times \sigma^2}\, S(\tau) \right] d\tau$$

$$+ 1 \times \sigma\, S(\tau)\, dW(\tau) = \mu S(\tau)\, d\tau + \sigma S(\tau)\, dW(\tau)$$

Then:

$$df(\tau, X(\tau)) = \left[\frac{\partial f}{\partial \tau} + \mu(X(\tau))\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X(\tau))\frac{\partial^2 f}{\partial x^2} \right] d\tau$$

$$+ \sigma(X(\tau)) \frac{\partial f}{\partial x} \, dW(\tau)$$

$$dX(\tau) = \mu X(\tau)\, d\tau + \sigma X(\tau)\, dW(\tau)$$

$$d(\underbrace{e^{-R\tau} X(\tau)}_{Y(\tau)}) = (\mu - R)\underbrace{e^{-R\tau} X(\tau)}_{Y(\tau)} d\tau + \sigma \underbrace{e^{-R\tau} X(\tau)}_{Y(\tau)} dW(\tau)$$

$\{X(\tau), \tau \geq 0\}$: Geometric Brownian motion, with drift μ and volatility σ

$\{e^{-R\tau} X(\tau), \tau \geq 0\}$: Geometric Brownian motion with drift $\mu - R$ and volatility σ

Theorem: A Itô's diffusion is a martingale with respect to $\{\mathcal{F}_\tau^W, \tau \geq 0\}$ if and only if the drift term is zero, i.e.:

$$dX(\tau) = \sigma(X(\tau))\, dW(\tau)$$

Returning to the previous example,

$$d(\underbrace{e^{-R\tau} X(\tau)}_{Y(\tau)}) = (\mu - R)\underbrace{e^{-R\tau} X(\tau)}_{Y(\tau)} d\tau + \sigma \underbrace{e^{-R\tau} X(\tau)}_{Y(\tau)} dW(\tau)$$

(which is related with the Black-Scholes formula, as we will see), $\{Y(\tau), \tau \geq 0\}$ will be a martingale if and only if $\mu = R$.

9. Use Ito's formula to compute $\mathbb{E}[W^4(t)]$.

$W(t) \sim \mathcal{N}(0,t)$

$E[W(t)] = 0$

$\text{Var}(W(t)) = E[W^2(t)] - \overbrace{E^2[W(t)]}^{=0} = t$

$\Rightarrow E[W^2(t)] = t$

$E[W^3(t)];\ E[W^4(t)] = ?\ ..$

Plan:

i) compute $dW^4(t)$

ii) compute its integral

iii) calculate its expected value; we will use the first Ito's isometry.

i) $dW^4(t) = \cancel{4 W^3(t)\,dt}$

$dX(t) = 0\,dt + 1\,dW(t)$

$f(t, x) = x^4 \equiv f(x)$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

Then:

$$df(t, X(t)) = \left[\frac{\partial f}{\partial t} + \mu(X(t))\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X(t))\frac{\partial^2 f}{\partial x^2}\right] dt$$

$$+ \sigma(X(t))\frac{\partial f}{\partial x}\, dW(t)$$

So: $dW^4(t) = \left[0 + 0 + \frac{1}{2} \times 1 \times 12W^2(t)\right] dt$

$+ \ 1 \ 4W^3(t)\,dW(t)$

$\Leftrightarrow \quad dW^4(t) = 6W^2(t)\,dt + 4W^3(t)\,dW(t)$

ii) $\underbrace{\int_0^t dW^4(s)}_{W^4(t) - W^4(0)} = \int_0^t 6W^2(s)\,ds + \int_0^t 4W^3(s)\,dW(s)$

$= W^4(t)$

$W^4(t) = \int_0^t 6W^2(s)\,ds + \underbrace{\int_0^t 4W^3(s)\,dW(s)}_{0}$

iii) $E[W^4(t)] = \int_0^t 6E[W^2(s)]ds + 0$ (by first Itô's isometry)

$= \int_0^t 6s\,ds = 6\frac{t^2}{2} = 3t^2$ //

9. Use Itô's formula to compute $E[W^4(t)]$.

10. Let $\{W(t), t \in \mathbb{R}^+\}$ denote the standard Brownian motion. Classify the following processes as martingales or not:

a) $X(t) = 2W(t) - 2$
b) $Y(t) = W^2(t) - t$
c) $Z(t) = t^2W(t) - 2\int_0^t sW(s)ds$

a) $\{X(t) = 2W(t) - 2,\ t \geq 0\}$

• by definition: $E[X(t) | \mathcal{F}_s^W] = X(s),\ \forall s \leq t$

$= E[2W(t) - 2 | \mathcal{F}_s^W] = 2E[W(t) | \mathcal{F}_s^W] -$

$- 2 = 2E[W(t) - W(s) + W(s) | \mathcal{F}_s^W] - 2$

$= 2E[W(t) - W(s) | \mathcal{F}_s^W] + 2E[W(s) | \mathcal{F}_s^W] - 2$

independence of increments

$= 2E[W(t) - W(s)] + 2W(s) - 2$

$= 2 \times 0 + 2W(s) - 2 = X(s)$ ✓

$E[W(t) - W(s)] = E[W(t)] - E[W(s)] = 0 - 0 = 0$

• by Itô's formula:

$d(2W(t) - 2) = d(2W(t)) - d(2)$

$= 2\,dW(t) = \underline{0\,dt} + 2\,dW(t)$

drift $= 0 \Rightarrow$ martingale!

Note:

- $d(a\,x(t)) = a\,dx(t)$
- $d(x(t)+a) = dx(t)$
- $d(x(t)y(t)) = dx(t)\,y(t) + x(t)\,dy(t)$
  $+ dx(t)\,dy(t)$

Homework: Prove that $d(e^{-rt}x(t)) =$

x : GBM $\qquad = (r-\mu)e^{-rt}x(t)\,dt + \sigma e^{-rt}x(t)\,dw(t)$

b) $\{Y(t) = w^2(t) - t\}$

$$d(w^2(t)) = d(w(t).w(t)) =$$

$$= w(t)\,dw(t) + dw(t)\,w(t) + \underbrace{dw(t)\,dw(t)}_{= dt}$$

$$= dt + 2w(t)\,dw(t)$$

$$\Rightarrow d(w^2(t) - t) = \cancel{dt} + 2w(t)\,dw(t) - \cancel{dt}$$
$$= 2w(t)\,dw(t)$$

and Thus $\{w^2(t) - t,\ t \geq 0\}$ is a martingale!

b) $Y(t) = W^2(t) - t$

c) $Z(t) = t^2W(t) - 2\int_0^t sW(s)ds$

$$dZ(t) = d(t^2w(t)) - d\big(\underbrace{2\int_0^t s\,w(s)\,ds}_{=}\big)$$
$$= d(t^2w(t)) - 2t\,w(t)\,dt$$

$$d(t^2w(t)) = ?$$

$dx(t) = 0\,dt + 1\,dw(t) \Rightarrow \mu(x(t)) = 0$
$\sigma(x(t)) = 1$

$f(t, x) = t^2 x$

$\frac{\partial f}{\partial t} = 2tx \qquad \frac{\partial f}{\partial x} = t^2 \qquad \frac{\partial^2 f}{\partial x^2} = 0$

$$d(t^2 w(t)) = \left[2t\,w(t) + 0 + 0\right] dt + t^2\,dw(t)$$
$$= 2t\,w(t)\,dt + t^2\,dw(t)$$

Therefore:

$$dZ(t) = \cancel{2t\,w(t)\,dt} + t^2\,dw(t) - \cancel{2t\,w(t)\,dt}$$
$$= t^2\,dw(t) \Rightarrow \text{it is a martingale!}$$

17. Let $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ be two stochastic processes satisfying the following system of SDE's:

$$dX(t) = \alpha X(t)dt + Y(t)dW(t)$$
$$dY(t) = \alpha Y(t)dt - X(t)dW(t)$$

with $X(0) = x_0$ and $Y(0) = y_0$, deterministic constants.

a) Show that $R = \{R(t), t \geq 0\}$, with $R(t) = X^2(t) + Y^2(t)$, is deterministic.

$$dR(t) = d(x^2(t)) + d(y^2(t)) = \ldots dt + 0\,dw(t)$$

$$d(x^2(t)) = d(x(t) \cdot x(t))$$

$$d(x(t)\,y(t)) = dx(t)\,y(t) + dy(t)\,x(t) + dx(t)\,dy(t)$$

$$d\,x^2(t) = 2x(t)\,dx(t) + (dx(t))^2$$
$$= 2\alpha x^2(t)\,dt + 2x(t)\,y(t)\,dw(t) +$$
$$+ (\alpha x(t)\,dt + y(t)\,dw(t))^2$$
$$= 2\alpha x^2(t)\,dt + 2x(t)\,y(t)\,dw(t) + .(dt)^2 + \ldots dt\,dw(t)$$
$$+ y^2(t)(dw(t))^2 =$$
$$= \left[2\alpha x^2(t) + y^2(t)\right] dt + 2x(t)\,y(t)\,dw(t)$$

$$dy^2(\tau) = 2y(\tau)\,dy(\tau) + (dy(\tau))^2$$
$$= 2\alpha y^2(\tau)\,d\tau - 2y(\tau)x(\tau)\,dw(\tau) + x^2(\tau)\,d\tau$$
$$= [2\alpha y^2(\tau) + x^2(\tau)]\,d\tau - 2y(\tau)x(\tau)\,dw(\tau)$$

Hence:

$$dR(\tau) = dx^2(\tau) + dy^2(\tau) =$$
$$= (x^2(\tau) + y^2(\tau))[2\alpha + 1]\,d\tau$$

ie: $\boxed{dR(\tau) = (2\alpha+1)R(\tau)\,d\tau}$ $\Rightarrow R(\tau) = [\ ]\,e^{(2\alpha+1)\tau}$

14. Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a twice differentiable function, such that the Ito's formula may be applied.

a) Show that $f(t, W(t))$ is a martingale if and only if

$$\frac{\delta}{\delta t} f(t,x) + \frac{1}{2} \frac{\delta^2}{\delta x^2} f(t,x) = 0, \quad \forall t, x$$

b) As an application, show that $\{W^2(t) - t, t \geq 0\}$ is a martingale. Show also this result from the definition of a martingale (the equality of the conditional expectation).

c) For $a, b \in \mathbb{R}$, define the process X by $\{X(t) = e^{aW(t) - bt}, t \geq 0\}$. Determine the relationship between a and b in order for X to be a martingale.

17. Let $X = \{X(t), t \geq 0\}$ and $Y = \{Y(t), t \geq 0\}$ be two stochastic processes satisfying the following system of SDE's:

$$dX(t) = \alpha X(t)dt + Y(t)dW(t)$$
$$dY(t) = \alpha Y(t)dt - X(t)dW(t)$$

with $X(0) = x_0$ and $Y(0) = y_0$, deterministic constants.

a) Show that $R = \{R(t), t \geq 0\}$, with $R(t) = X^2(t) + Y^2(t)$, is deterministic.

20. Consider the processes $\{X(t) : t \geq 0\}$ and $B(t) : t \geq 0\}$, where

$$
\begin{aligned}
X(t) &= W(t) - tW(1) \\
B(t) &= (t+1)X(\tfrac{t}{t+1})
\end{aligned}
$$

Show that $\{B(t), t \geq 0\}$ is also a Brownian motion.

22. Let $M = \{M(t), t \geq 0\}$, with

$$M(t) = \frac{1}{\sqrt{1-t}} \exp\left(-\frac{W^2(t)}{2(1-t)}\right)$$

a) Derive a stochastic differential equation such that M is a solution.

b) Show that M is a martingale.

c) Compute $\mathbb{E}[M(t)]$.